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Shape-Invariant Bounds and More General Estimates for Vector-Valued Elliptic–Parabolic Problems*

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1. INTRODUCTION

We derive estimates of the form

$$v(x) \in \psi(x) G \qquad (x \in \bar{\Omega}) \tag{1.1}$$

for functions $v: \bar{\Omega} \to \mathbb{R}^n$, where Ω is a bounded domain in $\mathbb{R}^m$, G denotes a closed and star-shaped set in $\mathbb{R}^n$, and ψ is a scalar-valued function to be determined. This function ψ has to satisfy certain differential inequalities which involve the image Mv of v under an elliptic (or parabolic) vector-valued differential operator M of the second order, and a real-valued function W describing the set G. In particular, v may be the solution of a problem $Mv = r$. More generally, estimates of the form

$$v(x) \in \psi_\ell(x)\, G_\ell \qquad (x \in \bar{\Omega};\ \ell = 1, 2, \ldots, N) \tag{1.2}$$

are considered.

Choosing $\psi(x) \equiv 1$ one obtains results on *invariant sets*. In general, however, the "size" of the bounding set $\psi(x)G$ in (1.1) may depend on x; only its "shape" remains invariant. We speak of *shape-invariant bounds* and *shape-invariant estimates*.

In Sections 3 through 5 estimates of the form (1.1) are discussed. The general method of proof is explained in Section 3 for the case of a semilinear differential operator

$$Mu(x) = \mathscr{L}[u](x) + f(x, u(x), Du(x)) \qquad \text{for} \quad x \in \Omega \tag{1.3}$$

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with uncoupled linear $\mathscr{L}[u]$, and Dirichlet boundary terms. Section 4 yields an application to linear operators M. In Section 5 more general boundary terms and quasilinear differential operators are treated; certain parabolic operators are included as a special case. Section 6 is concerned with estimates of the form (1.2). All these results have the form of *range domain implications*: $Mv \in C \Rightarrow v \in K$. Section 7 shows that under more restrictive conditions such results can be used to prove that a given problem $Mu = r$ has a solution v satisfying (1.1) or (1.2).

In deriving shape-invariant estimates (1.1) we assume that G has a smooth boundary. Then we must also assume that all components of M have the same *leading coefficients*. This disadvantage can be overcome by considering estimates of the form (1.2). For such estimates, however, the coupling of the components of Mv with respect to the derivatives of v, in general, must be weaker. These relations are discussed in Section 6.2. Pointwise norm bounds obtained with $W(y) = y^T y$ and two-sided bounds obtained with $2n$ functions W_i constitute extreme cases.

Statements on invariant sets for elliptic problems have been derived by several authors. In particular, the works of Amann [2], Lemmert [9, 10], Martin [11], Redheffer and Walter [14], Schmitt [15], and Weinberger [25] are more or less strongly related to the theory presented here. In these papers estimates of the form $v(x) \in G$ $(x \in \bar{\Omega})$ are proved for solutions v of boundary value problems with differential operators of the form (1.3) (with f not depending on the derivative Du in [11, 25]).

The results in these papers are essentially of two different types; call them (U) and (E). The results of type (U) state that no solution can have values outside G. Here the assumptions and theorems can often be used to prove a uniqueness statement, which however, is not the main object of the theory, in general. In results of type (E) the existence of a solution with values in G is established. Analogously, one can distinguish between results of type (U) and (E) for more general estimates of solutions.

In this paper, we are mainly interested in results of type (U). Results of this type were also proved in [9, 10, 14, 25]. The papers [2, 11, 15] contain statements of type (E). Notice, however, that the invariance result of Weinberger [25] is very closely related to the existence theory (see the remarks in Section 6.1).

Naturally, the two types of results for elliptic problems require different assumptions. For instance, in the existence proofs stronger smoothness conditions on the coefficients of $\mathscr{L}$ and the function f are required, as well as growth restrictions on $f(x, y, p)$ with respect to p. Moreover, in general, the results of type (E) involve only properties of $f(x, y, p)$ for y belonging to the (closed) set G, while, on the contrary, in results of type (U) values $f(x, y, p)$ are used for $y \notin G$, or y belonging to the closure of the complement of G.

The common feature of all these results on invariant sets are certain *tangency conditions* (*tangent conditions*), which most often assume the form

$$\mathbf{n}(y)f(x,y,p) \geqslant 0 \qquad \text{for} \quad \mathbf{n}(y)p = o \tag{1.4}$$

and $x \in \Omega$, $y \in \partial G$, with (the row vector) $\mathbf{n}(y)$ being an outer normal of G at y. In our paper such conditions are replaced by differential inequalities for ψ (or the N functions ψ_i), which for $\psi(x) \equiv 1$ and $Mv = o$ can be written as

$$Q(x,y,p) + \mathbf{n}(y)f(x,y,p) \geqslant 0 \qquad \text{for} \quad \mathbf{n}(y)p = o \tag{1.5}$$

where $Q(x,y,p)$ denotes a quadratic form in p. For convex G this form may be replaced by 0. In many cases the occurrence of this quadratic form allows f to be more strongly coupled with respect to p.

Of course, the various papers emphasize different points and contain more special results. For example, Weinberger [25] proves generalizations of the *strong* maximum principle, and Lemmert [10] uses rather weak smoothness conditions of the functions v considered, defining derivatives in a special way. Our object is to obtain results which allow one to actually compute reasonably good bounds ψ (or ψ_i), either by analytical calculations or numerical methods.

Many invariance statements for parabolic problems also contain assumptions of the form (1.4) (see Bebernes and Schmitt [4], Chueh *et al.* [5], Redheffer and Walter [13] for results of type (U); and Amann [2], Bebernes and Schmitt [4], Weinberger [25] for results of type (E)). Our results (of type (U)) on parabolic problems in Section 5 are obtained by simply generalizing corresponding results for elliptic problems, and hence they are proved with the same global method. In general, results of type (U) for parabolic problems are derived using a different method associated with the names of Nagumo and Westphal. For example, this method was applied for obtaining two-sided bounds (see Lakshmikantham and Leela [8], Szarski [22], Walter [24]). This method could also be used here. The relation between the methods of proof and the resulting assumptions have been discussed in [18, p. 251ff] for ordinary differential equations.

There is an extensive literature on invariance statements and other estimates for ordinary differential equations. For references and discussions see, for example, Gaines and Mawhin [7], Schmitt [15] and [18, 19].

2. Notation and Auxiliary Means

All vectors, matrices, functions, etc., which occur in this paper are supposed to be real-valued. $\mathbb{R}^{n,m}$ denotes the set of $n \times m$ matrices $A = (a_{jk})$. A^T is the transposed matrix; $A^H = \frac{1}{2}(A + A^T)$ for quadratic A. The maximal and minimal eigenvalues of a symmetric matrix A are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively. We write $A \geqslant_d B$ if $A - B$ is symmetric and positive semi-definite. A vector (n-vector) is an element of $\mathbb{R}^{n,1} = \mathbb{R}^n$, i.e., a column-vector. Null elements of vector and function spaces are usually denoted by o; for null matrices, in general, the symbol O is used. For any vector y, $\|y\|$ denotes the Euclidean norm.

For matrices $A, B \in \mathbb{R}^{n,m}$ we introduce an inner product $A \cdot B = \sum_{j,k} a_{jk} b_{jk}$. This quantity is equivalent to the trace $\operatorname{tr}(A^T B)$ of the matrix product $A^T B$. One verifies that $A \cdot (BC) = AC^T \cdot B = B^T A \cdot C$, provided the occurring terms are defined. Moreover, $A \cdot B \geqslant n \lambda_{\min}(A)\, \lambda_{\min}(B)$ for symmetric positive semi-definite $A, B \in \mathbb{R}^{n,n}$.

For a bounded domain $\Omega \subset \mathbb{R}^m$ with points $x = (x_k)$, boundary $\partial\Omega$ and closure $\bar{\Omega}$, the space of continuous mappings $\bar{\Omega} \to \mathbb{R}^n$ will be denoted by $C_0^n(\bar{\Omega})$. This means we consider this space as the product of n spaces $C_0(\bar{\Omega})$ and denote the elements by $u = (u_i)$ with $u(x) \in \mathbb{R}^n$, $u_i \in C_0(\bar{\Omega})$. Notation like $C_2^n(\Omega)$ are defined analogously. The natural order relation in $C_0^n(\bar{\Omega})$ will be denoted by $\leqslant$, so that for $u, v \in C_0^n(\bar{\Omega})$

$$
\begin{aligned}
u \leqslant v \quad &\Leftrightarrow \quad u(x) \leqslant v(x) \;\; (x \in \bar{\Omega}) \quad \Leftrightarrow \quad u_i \leqslant v_i \; (i = 1, 2, \ldots, n) \\
&\Leftrightarrow \quad u_i(x) \leqslant v_i(x) \; (x \in \bar{\Omega},\ i = 1, 2, \ldots, n).
\end{aligned}
$$

As in these relations we shall, in general, carefully distinguish between functions u and their values $u(x)$. In some places, however, the independent variable will be omitted or partly omitted, so that $f(x, u)$ may stand for $f(x, u(x))$, for example.

We denote derivatives by $D_k = \partial/\partial x_k$, $D_{jk} = D_j D_k$. Moreover, for $u \in C_1^n(\Omega)$, $Du(x)$ is the $n \times m$ matrix with elements $D_k u_i(x)$; and for $\varphi \in C_2(\Omega)$, $D^2\varphi(x)$ is the $m \times m$ matrix with elements $D_{jk}\varphi(x)$. For convenience, we write also $u'(x) = Du(x) \in \mathbb{R}^{n,m}$ and $\varphi''(x) = D^2\varphi(x) \in \mathbb{R}^{m,m}$.

We shall consider statements of the form $y \in \alpha G$, where $y \in \mathbb{R}^n$, $\alpha \geqslant 0$, and $G \subset \mathbb{R}^n$. We then will require the following:

Assumption ($\mathcal{g}$). G is a closed set in $\mathbb{R}^n$ with boundary Γ, $y = o$ is an interior point of G, and G is star-shaped with respect to this point. There exists a function $W \in C_0(\mathbb{R}^n)$ such that $W(y) = 1$ for $y \in \Gamma$, $W(y) < 1$ for y in the interior of G, and $W(y) > 1$ for $y \notin G$. W is twice continuously differentiable on the set H which consists of all $y \in \mathbb{R}^n$ such that $\alpha y \in \Gamma$ for some $\alpha > 0$. Moreover,

$$
W'(y)y > 0 \qquad \text{for} \quad y \in \Gamma. \tag{2.1}
$$

For $\alpha > 0$ the set αG is defined by $\alpha G = \{ y \in \mathbb{R}^n : \alpha^{-1} y \in G \}$. Moreover, $0G$ denotes the intersection of all αG with $0 < \alpha < \varepsilon$ and some $\varepsilon > 0$.

If G satisfies assumption $(\mathscr{g})$, the (generalized) Minkowski functional $V \in C_0(\mathbb{R}^n)$ is defined by

$$V(y) = \alpha^{-1} \quad \text{for} \quad y \in H \text{ with } \alpha y \in \Gamma, \qquad V(y) = 0 \quad \text{for} \quad y \notin H. \tag{2.2}$$

This function V, too, satisfies assumption $(\mathscr{g})$. Moreover, V is positive homogeneous, i.e., $V(ty) = tV(y)$ for $y \in \mathbb{R}^n$, $t \geqslant 0$; and $H = \{y: V(y) > 0\}$. Finally, for $y \in \mathbb{R}^n$ and $\alpha \geqslant 0$

$$y \in \alpha G \Leftrightarrow V(y) \leqslant \alpha. \tag{2.3}$$

The theory of Sections 3 through 5 could be generalized by considering less smooth functions W and hence also sets G with less smooth boundaries Γ. To formulate the weakest possible smoothness conditions, however, is not our object here. Many cases with less smooth boundary Γ can be treated with the theory in Section 6. This theory allows one also to consider sets G which are not star-shaped.

3. Semi-Linear Operators with Dirichlet Boundary Terms

3.1. *The Basic Theorem*

This section is concerned with operators M given by

$$Mu(x) = \mathscr{L}[u](x) + f(x, u(x), u'(x)) \qquad \text{for} \quad x \in \Omega, \tag{3.1}$$

$$Mu(x) = u(x) \qquad \text{for} \quad x \in \partial\Omega, \tag{3.2}$$

for functions $u \in R := C_0^n(\bar{\Omega}) \cap C_2^n(\Omega)$ on the closure $\bar{\Omega}$ of a bounded domain $\Omega \subset \mathbb{R}^m$. Here, f maps $\Omega \times \mathbb{R}^n \times \mathbb{R}^{n,m}$ into $\mathbb{R}^n$, and $\mathscr{L}[u](x)$ is an n-vector with components

$$\mathscr{L}_i[u](x) = L[u_i](x) \qquad (i = 1, 2,..., n),$$

where for $\varphi \in \mathscr{R} := C_0(\bar{\Omega}) \cap C_2(\Omega)$ and $x \in \Omega$

$$L[\varphi](x) = -A(x) \cdot \varphi''(x) + b(x) \cdot \varphi'(x) \tag{3.3}$$

with given $A(x) \in \mathbb{R}^{m,m}$, $b(x) \in \mathbb{R}^{1,m}$, such that $A(x) = A^T(x) \geqslant_d O$.

We are interested in estimates of the form

$$v(x) \in \psi(x)G \qquad (x \in \bar{\Omega})$$

for a function $v \in R$ by a function $\psi \geqslant o$ in $\mathscr{R}$, where $G \subset \mathbb{R}^n$ denotes a given set which satisfies assumption ($\mathcal{g}$) in Section 2. In particular, G is described by a function $W \in C_0(\mathbb{R}^n)$, as required in this assumption. In the following theoretical results we consider v and ψ as fixed given functions, although in applications v will be unknown and ψ, in general, will have to be constructed.

Finally, let $\{\mathfrak{z}_\lambda : 0 \leqslant \lambda < \infty\}$ denote a given family of functions in $\mathscr{R}$ such that $\mathfrak{z}_\lambda(x)$ is continuous on $[0, \infty) \times \bar{\Omega}$, $\mathfrak{z}_0 = o$,

$$\mathfrak{z}_\lambda(x) > 0 \qquad \text{for} \quad \lambda > 0 \text{ and } x \in \partial\Omega, \tag{3.4}$$

and the function $\psi_\lambda = \psi + \mathfrak{z}_\lambda$ have the following properties:

$$\psi_\lambda(x) > 0 \qquad \text{for} \quad \lambda > 0 \text{ and } x \in \bar{\Omega},$$

$$\psi_\lambda(x) \to \infty \qquad \text{for} \quad \lambda \to \infty, \text{ uniformly in } x \in \bar{\Omega}.$$

THEOREM 1. *Suppose that the following conditions are satisfied.*

($\mathcal{j}$) *The inequality*

$$\omega(\eta)\, L[\psi_\lambda](x) + Q(x, \eta, q)\, \psi_\lambda(x) + W'(\eta)\, f(x, \psi_\lambda(x)\eta, \psi_\lambda(x)q + \eta\psi'_\lambda(x)) \\ > W'(\eta)\, Mv(x) \tag{3.5}$$

with

$$\omega(\eta) = W'(\eta)\eta \qquad \text{and} \qquad Q(x, \eta, q) = A(x) \cdot q^T W''(\eta) q$$

holds for all $x \in \Omega$, $\lambda > 0$, $\eta \in \mathbb{R}^n$, $q \in \mathbb{R}^{n,m}$ which satisfy

$$W(\eta) = 1, \qquad W'(\eta)q = o; \tag{3.6}$$

$$v(x) = \psi_\lambda(x)\eta, \qquad v'(x) = \eta\psi'_\lambda(x) + \psi_\lambda(x)q. \tag{3.7}$$

($\mathcal{k}$) $v(x) \in \psi(x)G$ *for all $x \in \partial\Omega$.*

Then

$$v(x) \in \psi(x)G \qquad \textit{for all} \quad x \in \bar{\Omega}. \tag{3.8}$$

Proof. In this proof, we shall assume that $b(x) = o$ for $x \in \Omega$, without loss of generality. (Incorporating $b(x) \cdot u'_i(x)$ into $f_i(x, u(x), u'(x))$ yields the same results, due to the side condition (3.6).)

According to (2.3), the estimate (3.8) can be written as $V(v(x)) \leqslant \psi(x)$ for all $x \in \bar{\Omega}$, where V is defined by (2.2). There exists a minimal value $\lambda \geqslant 0$ such that $V(v(x)) \leqslant \psi_\lambda(x)$ $(x \in \bar{\Omega})$. Since for $\lambda = 0$ this inequality is equivalent to the statement of the theorem, we assume that $\lambda > 0$. Then $V(v(\xi)) = \psi_\lambda(\xi) > 0$ for some $\xi \in \bar{\Omega}$. Due to assumption ($\mathcal{k}$) and (3.4) the point ξ cannot belong to $\partial\Omega$. Thus $\xi \in \Omega$, and $V(v(x)) > 0$ for all x in a suitable neighborhood of ξ. In this neighborhood we can write

$$v(x) = \rho(x)\,\eta(x) \qquad \text{with} \quad W(\eta(x)) = 1 \text{ and } \rho(x) = V(v(x)). \qquad (3.9)$$

Differentiating twice we then obtain the following relations where the variable x is omitted, for convenience:

(i) $W(\eta) = 1$,

(ii) $W'(\eta)\,\eta' = o$,

(iii) $(D_j\eta)^T\, W''(\eta)\, D_k\eta + W'(\eta)\, D_{jk}\eta = 0$ $(j, k = 1, 2,\ldots, m)$,

(α) $v = \rho\eta$,

(β) $v' = \rho\eta' + \eta\rho'$,

(γ) $D_{jk}v = \eta D_{jk}\rho + D_k\rho D_j\eta + D_j\rho D_k\eta + \rho D_{jk}\eta$ $(j, k = 1, 2,\ldots, m)$.

In particular, the above equations hold at $x = \xi$.

Since $\psi_\lambda - \rho$ assumes a relative minimum at ξ, we obtain, in addition:

(a) $\rho(\xi) = \psi_\lambda(\xi)$,

(b) $\rho'(\xi) = \psi_\lambda'(\xi)$,

(c) $\rho''(\xi) \leqslant_d \psi_\lambda''(\xi)$.

In the remainder of the proof, all the functions that occur need only be considered at the point ξ. We shall omit this sign, for simplicity, so that A stands for $A(\xi)$, etc.

Since $A \geqslant_d O$, one derives from (c) that (at $x = \xi$)

(d) $L[\rho] \geqslant L[\psi_\lambda]$.

Applying $W'(\eta)$ to (γ) and using (ii) and (iii) one obtains

$$W'(\eta)(D_{jk}v) = \omega(\eta)\, D_{jk}\rho - \rho(D_j\eta)^T\, W''(\eta)\, D_k\eta.$$

Multiplying this relation by $-a_{jk}$ and then adding over j and k one arrives at

$$W'(\eta)\,\mathscr{L}[v] = \omega(\eta)\, L[\rho] + Q(x, \eta, q)\rho \qquad \text{with} \quad q = \eta'.$$

In this equation one replaces $\mathscr{L}[v]$ by $Mv - f(\xi, v, v')$ and then eliminates v and v' in $f(\xi, v, v')$ applying (α), (β). Finally, using the relations (a), (b) and (d) one obtains an inequality which contradicts the required differential inequality (3.5) at $x = \xi$ with $\eta = \eta(\xi)$ and $q = \eta'(\xi)$. Consequently, $\lambda = 0$, and hence (3.8) holds. ∎

If G is convex, then the term $Q(x, \eta, q)$ in (3.5) may be replaced by 0. More precisely, we obtain a sufficient condition for (3.5), if we replace

$Q(x, \eta, q)$ by $\lambda_{\min}(H^T W''(\eta)H)\,\lambda_{\min}(A(x))q \cdot q$, where $H = H(\eta)$ denotes an $n \times (n-1)$ matrix such that $W'(\eta)H = o$ and $H^T H$ is the $(n-1) \times (n-1)$ unit matrix.

EXAMPLE 1. For the case $W(y) = y^T y$ one obtains

$$W'(\eta) = 2\eta^T, \qquad W''(\eta) = 2I, \qquad \omega(\eta) = 2 \qquad \text{for} \quad W(\eta) = 1,$$
$$Q(x, \eta, q) = 2\,\mathrm{tr}(qA(x)\,q^T) \geqslant 2\lambda_{\min}(A(x))q \cdot q.$$

3.2. *Discussion of the Results*

We shall now discuss the assumptions of the theorem.

(a) First we notice that differential inequalities for all functions $\psi_\lambda = \psi + \mathfrak{z}_\lambda$ with $\lambda > 0$ are required. Since $\psi = \psi_0$ is the function which we want to determine explicitly, we may try to replace each inequality (3.5) for ψ_λ by a differential inequality for ψ which does not contain the parameter λ, and an additional assumption involving $\mathfrak{z}_\lambda$. This can be achieved by a simple splitting. Inequality (3.5) has the form $\mathscr{F}_M(x, \eta, q, \psi_\lambda) > W'(\eta)\,Mv(x)$ with a suitable mapping $\mathscr{F}_M$ which depends on the given operator M. Obviously, this inequality is satisfied if $\mathscr{F}_M(x, \eta, q, \psi) \geqslant W'(\eta)\,Mv(x)$ and

$$\mathscr{F}_M(x, \eta, q, \psi + \mathfrak{z}_\lambda) > \mathscr{F}_M(x, \eta, q, \psi). \tag{3.10}$$

The first of these two inequalities is equivalent to

$$\omega(\eta)\,L[\psi](x) + Q(x, \eta, q)\,\psi(x) + W'(\eta)\,f(x, \psi(x)\eta, \psi(x)q + \eta\psi'(x)) \geqslant W'(\eta)\,Mv(x). \tag{3.11}$$

Now the idea is to construct a family $\{\mathfrak{z}_\lambda\}$ which satisfies (3.10) for a certain class of operators M (where ψ is arbitrary or subject to some unessential restrictions). Then, for an operator M in this class one need only solve the differential inequalities (3.11) for ψ.

This procedure was applied to several examples of nonlinear ordinary differential operators and an inner product $W(y) = \langle y, y\rangle$ in [18]. The methods used there can be carried over to the more general case considered here. In Section 4 we use such a splitting for linear operators.

We point out, however, that for a given operator M this general procedure need not be optimal. For example, if one knows certain properties of v a priori, it may be better to first exploit the side conditions (3.7) and then to split the resulting inequality (see [18, p. 314ff.], where the a priori inequality $v_2(x) \leqslant 0$ was used; also, see Section 5.3).

The parameter $\lambda \in [0, \infty)$ was introduced to insure that the results hold without any restrictions on the "size" of v. If, however, certain properties of

v are known which imply that $v(x) \in \psi_{\lambda_0}(x)G$ $(x \in \bar{\Omega})$ for a certain value $\lambda_0 < \infty$, then the functions $\mathfrak{z}_\lambda$ need only be given for $\lambda \leqslant \lambda_0$.

(b) Both inequalities (3.5) and (3.11) contain parameters $\eta \in \mathbb{R}^n$ and $q \in \mathbb{R}^{n,m}$ which are restricted only by the side conditions (3.6), (3.7). First observe that *these parameters* η *and* q *may be replaced by the values* $\eta(x)$ *and* $\eta'(x)$ *of the function* η *defined by* (3.9), *so that one may use*

$$\eta = (V(v(x))^{-1} v(x), \qquad q = d/dx[(V(v(x))^{-1} v(x)] \tag{3.12}$$

in (3.5).

Thus, if one knows some properties of v, these relations may enable one to further restrict the values of parameters η, q for which (3.5) is to be required. There seems to be no practical way, however, to completely avoid the occurrence of the parameters η, q, except by further estimates or by requiring additional assumptions on M. For example, in the corresponding theory on two-sided bounds only parameters occur which correspond to η, if one requires that M be weakly coupled. If in this case, M is also quasimonotone, *all* parameters can be eliminated. (See Section 6.3.)

For a bounded set G the parameter η assumes values in the bounded set Γ. The set of parameters q, however, is not bounded (unless (3.12) can be used). Therefore, the differential inequalities (3.5) and (3.11) impose considerable restrictions on the way in which $f(x, y, p)$ may depend on p. For example, in the corresponding results on two-sided bounds, where G is a cube, these restrictions essentially mean that f *must* only be weakly coupled. However, if the values of Q that occur are strictly positive, then this assumption is not necessary. (Compare the discussion of linear operators in Section 4.)

(c) The choice $\psi(x) \equiv 1$ yields results on invariant sets. Such a result was proved by Weinberger [25] using condition (1.4). The corresponding Theorem 3 in [25], however, has a different form, more closely related to the existence theory (see the remarks in Section 7.1). For $\psi(x) \equiv 1$, the differential inequality (3.11) assumes the form (1.5). In condition (3.10) one may then use $\mathfrak{z}_\lambda(x) \equiv \lambda$ or a more general family of functions $\mathfrak{z}_\lambda$.

The essential assumptions of the theory on invariant sets of Lemmert [9, 10] and Redheffer and Walter [14], also consist of two conditions, called *uniqueness condition* and *tangent condition* in [14]. The uniqueness condition essentially is a generalized Lipschitz condition for f involving a (possibly nonlinear) *uniqueness function* w; the tangent condition is an inequality involving $\mathbf{n}f$. For example, in [9] an inequality of the form

$$\mathbf{n}(P(y))\, f(x, y, p) \geqslant w(x, \| y - P(y)\|, \mathbf{n}(P(y))p)$$

is required for $y \notin G$, where $P(y)$ denotes the projection of y on G. For

$y \to P(y) \in \Gamma$ this inequality assumes the form (1.4). In [14] a Lipschitz condition for $f(x, y, p)$ with respect to p is required and, in addition, an inequality of the form

$$\mathbf{n}(y) f(x, y + \lambda \mathbf{n}(y), p) \geqslant 0 \qquad \text{for} \quad y \in \Gamma,\ \lambda > 0,\ \mathbf{n}(y)p = o. \quad (3.13)$$

The relation between estimates of f by uniqueness functions and the existence of suitable families $\{\mathfrak{z}_\lambda\}$ has been illustrated in [18, pp. 172, 173, 253] for the case of two-sided bounds for ordinary differential operators of the first order. Our results are derived using a continuity principle described in [17; 18, p. 234ff.], which involves a family of sets K_λ. The above proof corresponds to sets K_λ which consist of all $u \in R$ such that $u(x) \in (\psi(x) + \mathfrak{z}_\lambda(x))G$ $(x \in \bar{\Omega})$. For a convex set G one could also use the continuity principle with K_λ denoting the set of all $u \in R$ such that $u(x)$ has at most the distance $\mathfrak{z}_\lambda(x)$ from $\psi(x)G$ $(x \in \bar{\Omega})$. Then for $\psi(x) \equiv 1$ and $\mathfrak{z}_\lambda(x) \equiv \lambda$ terms such as $y + \lambda \mathbf{n}(y)$ would occur (compare (3.13)). In [20] we reported briefly on the result in Theorem 1 under more special assumptions, without giving a detailed proof.

4. Linear Operators

The results of the preceding section will here be applied to linear operators M, in order to illustrate under which conditions the assumptions of Theorem 1 can be satisfied. We consider the special case where

$$W(y) = y^T y \qquad \text{and} \qquad A(x) \text{ is positive definite for } x \in \Omega, \quad (4.1)$$

and define $\|y\| = (y^T y)^{1/2}$. Similar results can be obtained for a more general function W with positive definite second derivative. The methods used can often be applied to nonlinear problems also (see Section 5.3).

Suppose that

$$f(x, u(x), u'(x)) = \sum_{j=1}^{n} B_j(x)(Du_j(x))^T + C(x)\, u(x) \quad (4.2)$$

with $B_j(x) \in \mathbb{R}^{n,m}$ and $C(x) = \mathbb{R}^{n,n}$. The sum involving the terms B_j can also be written in the form

$$\sum_{k=1}^{m} \mathscr{B}_k(x)(D_k u(x)) \qquad \text{with} \quad \mathscr{B}_k(x) \in \mathscr{R}^{n,n}.$$

We choose a family $\mathfrak{z}_\lambda = \lambda z$ with $z \in \mathscr{R}$ satisfying $z(x) > 0$ $(x \in \bar{\Omega})$. Then the inequalities (3.10) and (3.11) are equivalent to $\mathscr{F}_M(x, \eta, q, z) > 0$ and

$\mathcal{F}_M(x, \eta, q, \psi) \geqslant W'(\eta)\, Mv(x)$, respectively. We shall discuss conditions on M such that suitable functions z and ψ exist. Here, the assumptions on z assume the form

$$-A \cdot z'' + \sum_{k=1}^{m} \eta^T(\mathcal{B}_k + b_k I)\,\eta(D_k z) + cz > 0 \qquad \text{on } \Omega \tag{4.3}$$

with $c = \sum_{i=1}^{n} c_i + \eta^T C \eta$, $c_i = q_i A q_i^T + \eta^T B_i q_i^T$ and q_i the ith row of $q \in \mathbb{R}^{n,m}$. For $\eta^T \eta = 1$ and $\eta^T q = o$ one calculates $c = c(x, \eta, q) \geqslant s(x, \eta)$ with

$$s = \eta^T \left(C - \tfrac{1}{4} \sum_{i=1}^{n} B_i A^{-1} B_i^T \right) \eta + \tfrac{1}{4} l^T A^{-1} l, \qquad l^T = \eta^T \sum_{i=1}^{n} \eta_i B_i. \tag{4.4}$$

Using these relations together with other simple estimates of the terms in (4.3), as well as corresponding estimates for ψ, we arrive at the following result, where for $\varphi \in \mathcal{R}$ the term $\mathcal{M}\varphi$ is defined by

$$\mathcal{M}\varphi(x) = -A(x) \cdot \varphi''(x) - \tau(x)\, \|\varphi'(x)\| + \sigma(x)\, \varphi(x) \qquad \text{for} \quad x \in \Omega$$

$$\mathcal{M}\varphi(x) = \varphi(x) \qquad \text{for} \quad x \in \partial\Omega$$

with

$$\sigma(x) = \inf\{s(x, \eta) \colon \|\eta\| = 1\}, \qquad \tau(x) = \left(\sum_{k=1}^{m} \|\mathcal{B}_k^H(x) + b_k(x) I\|^2 \right)^{1/2},$$

and $\sigma(x)$ may be replaced by any lower bound, such as

$$\sigma_0(x) = \lambda_{\min}(C^H(x)) - \tfrac{1}{4} \lambda_{\max} \left(\sum_{i=1}^{n} B_i(x)\, A^{-1}(x)\, B_i^T(x) \right). \tag{4.5}$$

THEOREM 2. *If there exists a function $z \in \mathcal{R}$ such that $z(x) > 0$ $(x \in \bar{\Omega})$ and $\mathcal{M}z(x) > 0$ $(x \in \Omega)$, then for each $v \in R$ and each $\psi \geqslant o$ in $\mathcal{R}$*

$$\|Mv(x)\| \leqslant \mathcal{M}\psi(x) \;\; (x \in \bar{\Omega}) \Rightarrow \|v(x)\| \leqslant \psi(x) \;\; (x \in \bar{\Omega}). \tag{4.6}$$

Inequalities of the type $\mathcal{M}z(x) > 0$ have been discussed in the theory of inverse-positive differential operators. For example, in [16] a function

$$z(x) = h(r) := \int_r^{r_0} s \exp\left[\int_0^s \rho(t)\, dt \right] ds \tag{4.7}$$

was used, where $r = \|x - x_0\|$, $x_0 \in \Omega$, $r < r_0$ for $x \in \bar{\Omega}$, $\rho \in C_0[0, r_0]$ and $\rho(r) \geqslant 0$ $(0 \leqslant r \leqslant r_0)$. With this function one obtains here the following statement.

COROLLARY 2a. *Implication* (4.6) *holds if*

$$A(x) \cdot I + \sigma(x)\, h(0) > 0 \tag{4.8}$$

and $\rho(r)\, \lambda_{\min}(A(x)) \geqslant \tau(x)$ *for* $x \in \Omega$ *and* $r = \|x - x_0\|$.

For constant $\rho(r) \equiv \rho \geqslant 0$ *and a suitable* x_0 *condition* (4.8) *is satisfied, if*

$$A(x) \cdot I + \tfrac{1}{8}\sigma(x)\, d^2 \exp(\tfrac{1}{2}\rho d) > 0, \tag{4.9}$$

where d is any number greater than the diameter of Ω.

The above estimates can be improved in various ways. For example, one may try to take into account the signs of the derivatives $D_k z$ (see Example 3). The above formulas show, however, that *the operator M may be strongly coupled.*

EXAMPLE 2. Let $n = m = 2$, $A(x) \equiv I$ and

$$B_1 = \begin{pmatrix} 0 & 0 \\ \gamma & \delta \end{pmatrix}, \quad B_2 = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}, \quad \text{so that } \mathscr{B}_1 = \begin{pmatrix} 0 & \alpha \\ \gamma & 0 \end{pmatrix}, \mathscr{B}_2 = \begin{pmatrix} 0 & \beta \\ \delta & 0 \end{pmatrix},$$

with given numbers α, β, γ, δ and assume, for simplicity, that b_1, b_2 and C are also constant. Using (4.9) with $\sigma(x)$ replaced by $\sigma_0(x)$ in (4.5), we obtain the following sufficient condition for (4.8):

$$\lambda_{\min}(C^H) > \tfrac{1}{4} \max(\alpha^2 + \beta^2, \gamma^2 + \delta^2) - 16 d^{-2} \exp(-\tfrac{1}{2}\rho d).$$

with $\rho = [(|b_1| + \frac{1}{2}|\alpha + \gamma|)^2 + (|b_2| + \frac{1}{2}|\beta + \delta|)^2]$. This inequality is always satisfied for sufficiently small d. ∎

EXAMPLE 3. Consider the same case as in the previous example, but assume now that the matrices $\mathscr{B}_1 + b_1 I$ and $-(\mathscr{B}_2 + b_2 I)$ are positive semidefinite. Conditions of this type can be taken into consideration by choosing z such that the derivatives $D_k z$ have suitable signs. In the present case we choose z as in (4.7). Now, however, x_0 denotes a point outside Ω such that Ω lies in the fourth quadrant of the coordinate system described by $\tilde{x} = x - x_0$. For $\rho = 0$, condition (4.8) is now satisfied, if

$$2 + [\lambda_{\min}(C^H) - \tfrac{1}{4} \max(\alpha^2 + \beta^2, \gamma^2 + \delta^2)]\, \tfrac{1}{2}(r_0^2 - r_1^2) > 0$$

with r_1 denoting the minimum of all $\|x - x_0\|$ with $x \in \bar{\Omega}$. Observe that the numbers b_1 and b_2 do not occur in this formula. ∎

The above results contain generalizations of the *boundary maximum principle*, such as the following one:

COROLLARY 2b. *Suppose that* $\lambda_{\min}(A(x)) \geqslant \alpha_1 > 0$, $\tau(x) \leqslant \alpha_2$, $\sigma(x) \geqslant 0$ *for* $x \in \Omega$ *with constants* α_1, α_2. *Then for each* $v \in R$ *satisfying* $Mv(x) = o$ $(x \in \Omega)$ *and each constant* $\kappa \geqslant 0$

$$\|v(x)\| \leqslant \kappa \ (x \in \partial\Omega) \Rightarrow \|v(x)\| \leqslant \kappa \ (x \in \bar{\Omega}). \tag{4.10}$$

Statements of this type (4.10) have also been considered by Szeptycki [23], Styś [21] and Miranda [12]. In [21, 23] it is assumed that $B_i = O$ for all i and that C is positive definite or positive semi-definite, respectively. Miranda [12] makes an assumption which is equivalent to $s(x, \eta) \geqslant c_0 > 0$ $(x \in \Omega,\ \|\eta\| = 1)$ with s defined by (4.4). However, this assumption is formulated differently, using the matrices $\mathscr{B}_k$ instead of the matrices B_i.

5. MORE GENERAL OPERATORS

5.1. *More General Boundary Terms*

We shall now generalize the theory of Section 3 in such a way that also problems with boundary conditions more general than Dirichlet conditions can be treated and, in particular, certain parabolic problems are included. We assume, as before, that $\Omega \subset \mathbb{R}^m$ is a bounded domain and that for $x \in \Omega$ the term $Mu(x)$ is defined by (3.1). For $x \in \partial\Omega$, however, $Mu(x)$ may contain derivatives with respect to certain local coordinates $\nu, \mu_1, \mu_2, ..., \mu_r$:

$$Mu(x) = \bar{\mathscr{L}}[u](x) + g(x, u(x), u_\mu(x)) \qquad \text{for} \quad x \in \partial\Omega, \tag{5.1}$$

where $g(x, y, p) \in \mathbb{R}^n$ is defined for $x \in \partial\Omega$, $y \in \mathbb{R}^n$, $p \in \mathbb{R}^{n,r}$ and the n-vector $\bar{\mathscr{L}}[u](x)$ has the components

$$\bar{\mathscr{L}}_i[u](x) = \bar{L}[u_i](x) \tag{5.2}$$

with

$$\bar{L}[\varphi](x) = -\bar{A}(x) \cdot \varphi_{\mu\mu}(x) - \alpha(x)\,\varphi_\nu(x) + \beta(x) \cdot \varphi_\mu(x) + \gamma(x)\,\varphi(x) \tag{5.3}$$

for scalar-valued φ.

Here, lower indices ν and μ denote derivatives, $\varphi_\mu(x)$ is the row r-vector with components $\partial/\partial\mu_j\varphi(x)$, $\varphi_{\mu\mu}(x)$ is the $r \times r$ matrix with elements $\partial^2/\partial\mu_j\partial\mu_k\varphi(x)$, and the derivatives u_ν, u_μ are defined componentwise. The quantities ν, μ and r may depend on x. We assume that

$$\bar{A}(x) = \bar{A}^T(x) \in \mathbb{R}^{r,r}, \qquad \bar{A}(x) \geqslant_d O, \qquad \alpha(x) \in \mathbb{R},$$
$$\alpha(x) \geqslant 0, \qquad \beta(x) \in \mathbb{R}^{1,r}, \qquad \gamma(x) \in \mathbb{R},$$

and that

$$Mu(x) = u(x) \quad \text{if} \quad x \in \partial\Omega \quad \text{and } \bar{A}(x) = O, \quad \alpha(x) = 0, \quad \beta(x) = o. \tag{5.4}$$

We need to formulate some more conditions involving ν and μ_j. The idea is that (for a given $x \in \partial\Omega$) ν denotes a direction pointing from x into Ω and that the μ_j are tangential coordinates in x. We shall, however, not explicitly formulate smoothness conditions on $\partial\Omega$, u and φ such that the terms in (5.1) and (5.3) are meaningful, but use an implicit characterization instead.

Suppose that $\mathscr{R}$ denotes a linear subspace of $C_0(\bar{\Omega})$, such that for $\varphi \in \mathscr{R}$ the derivatives $\varphi_\nu(x)$, $\varphi_\mu(x)$, $\varphi_{\mu\mu}(x)$ are defined (as linear operations), if the corresponding terms $u_\nu(x)$, $u_\mu(x)$ and $u_{\mu\mu}(x)$ occur in the definition of $Mu(x)$. (For example, if $\alpha(x) \neq 0$, we may define $\varphi_\nu(x)$ to be a directional derivative in the usual sense.) We then define $R = \mathscr{R}^n$ and we assume that for $\varphi \in \mathscr{R}$ the relations

$$\varphi(x) \geqslant 0 \quad \text{for all } x \in \bar{\Omega}, \qquad \varphi(\xi) = 0 \quad \text{for some } \xi \in \partial\Omega$$

imply that

$$\varphi_\nu(\xi) \geqslant 0, \qquad \varphi_\mu(\xi) = o, \qquad \varphi_{\mu\mu}(\xi) = (\varphi_{\mu\mu}(\xi))^T \geqslant_d O,$$

if the occurring derivatives exist.

This property allows us to prove Theorem 3 below for operators M defined by (3.1) and (5.1). Again, $v \in R$ and $\psi \in \mathscr{R}$ denote fixed functions with $\psi \geqslant o$, and $\mathfrak{z}_\lambda \in \mathscr{R}$ denote functions as described in Section 3.

THEOREM 3. *Suppose that assumption* ($\mathscr{j}$) *of Theorem* 1 *and the following assumption* ($\mathscr{b}'$) *hold.*

($\mathscr{b}'$) *For each* $x \in \partial\Omega$ *the inequality*

$$\omega(\eta)\, \bar{L}[\psi_\lambda](x) + \bar{Q}(x, \eta, q)\, \psi_\lambda(x) + W'(\eta)\, g(x, \psi_\lambda(x)\eta, \psi_\lambda(x)q + \eta(\psi_\lambda)_\mu(x)) > W'(\eta)\, Mv(x)$$

with $\omega(\eta) = W'(\eta)\eta$ *and* $\bar{Q}(x, \eta, q) = \bar{A}(x) \cdot q^T W''(\eta) q$ *is satisfied for all* $\lambda > 0$, $\eta \in \mathbb{R}^n$, $q \in \mathbb{R}^{n,r}$ *such that* $W(\eta) = 1$, $W'(\eta)q = o$ *and*

$$v(x) = \psi_\lambda(x)\eta, \; v_\mu(x) = \psi_\lambda(x)q + \eta(\psi_\lambda)_\mu(x). \tag{5.5}$$

Then $v(x) \in \psi(x)G$ *for* $x \in \bar{\Omega}$.

We omit the proof of the theorem since this proof uses essentially the same ideas as that of Theorem 1. Moreover, the application of Theorem 3 is analogous to that of Theorem 1. For example, one may treat linear operators of the form (3.1), (5.1) using the methods explained in Section 4.

For points $x \in \partial\Omega$ with $Mu(x) = u(x)$ (as in (5.4)), assumption (ℓ') may be replaced by the sufficient condition $v(x) \in \psi(x)G$.

5.2. Quasilinear and More General Nonlinear Operators

The formal generalization of the results in Sections 3 and 5.1 to quasilinear operators presents no difficulty.

Suppose, for example, that $\mathscr{L}[u]$ in (3.1) has the components

$$\mathscr{L}_i[u](x) = -A(x, u(x), u'(x)) \cdot u_i''(x)$$

with symmetric $A(x, y, p) \in \mathbb{R}^{m,m}$ for $x \in \Omega$, $y \in \mathbb{R}^n$, $p \in \mathbb{R}^{n,m}$. Define then L by (3.3) with $A(x) := A(x, v(x), v'(x))$ and $b(x) = o$ (without loss of generality). *In this case Theorem* 1 *remains true, if* $A(x) \geqslant_d O$ *for* $x \in \Omega$. Of course, the side conditions (3.7) may be used to eliminate $v(x)$ and $v'(x)$ in $A(x, v(x), v'(x))$.

In an analogous way, Theorem 3 can be generalized. One may consider operators M such that

$$\bar{\mathscr{L}}_i[u](x) = -\bar{A}(x, u(x), u_\mu(x)) \cdot (u_i)_{\mu\mu}(x) - \alpha(x, u(x), u_\mu(x))(u_i)_v(x).$$

Then one defines $\bar{L}$ by (5.3) with $\bar{A}(x) = \bar{A}(x, v(x), v_\mu(x))$, $\alpha(x) = \alpha(x, v(x), v_\mu(x))$, $\beta = o$, $\gamma = 0$. *For the case so described, Theorem* 3 *remains true, if* $A(x) \geqslant_d O$ $(x \in \Omega)$ and $\bar{A}(x) \geqslant_d O$, $\alpha(x) \geqslant 0$ $(x \in \partial\Omega)$.

The ideas of the proofs can also be used in treating more general nonlinear problems. For example, consider an operator M with components

$$(Mu)_i(x) = F(x, u(x), u'(x), u_i''(x)) \qquad \text{for} \quad x \in \Omega.$$

Then the condition $A(x, v(x), v'(x)) \geqslant_d O$ has to be replaced by the requirement that $F(x, v(x), v'(x), S)$ is an antitonic function of $S \in \mathbb{R}^{m,m}$ with respect to the order relation $\leqslant_d$. For such problems it can sometimes be useful to employ also side conditions which involve the second derivative of v. Such conditions, which have the form of differential inequalities, can be derived from (α), (β), (γ), (c) in the proof of Theorem 1. (For instance, such conditions have been used in [17] for obtaining two-sided bounds for the Monge–Ampère equation.)

5.3. An Example

We consider a simple parabolic problem

$$
\begin{aligned}
u_t - \Delta u + \sum_{i=1}^{n} u_i(\partial/\partial\xi_i u) &= o && \text{on} \quad \Omega_0 x(0, T]; \\
u(\xi, t) &= o && \text{for} \quad \xi \in \partial\Omega_0, 0 < t \leqslant T; \\
u(\xi, 0) &= s(\xi) && \text{for} \quad \xi \in \bar{\Omega}_0,
\end{aligned}
\tag{5.6}
$$

where Ω_0 denotes a bounded domain in $\mathbb{R}^n$, and Δ is the Laplacian operator with respect to $\xi \in \mathbb{R}^n$. This problem (which was also treated by Amann [2]) has the same nonlinearity as the Navier–Stokes equations, but not the structural difficulties of those equations. The problem can be written as $Mu = r$ with an operator M as defined in this section, $m = n + 1$, $\Omega = \Omega_0 \times (0, T)$, $x_i = \xi_i$ $(i = 1, 2,..., n)$, $x_m = t$. For $\xi \in \Omega_0$ and $t = T$ the given differential equation is to be considered as a boundary condition $Mu(x) = o$ of the general form described above (choose $\mu = \xi$ and $\nu = -t$ in (5.3)). Let $R = C_0^n(\bar{\Omega}) \times C_2^n(\Omega_0 \times (0, T])$.

We want to obtain an estimate for a solution $v \in R$ of the given problem by applying Theorem 3 to a convex set G. For example, G may be the unit ball in $\mathbb{R}^n$, or the set of all $y \in \mathbb{R}^n$ with $y_1 \geqslant -1$. We choose $\psi_\lambda = \psi + \lambda z$, $\psi(x) = \varphi(\xi) \exp(-\kappa t)$, $z(x) = \exp(\delta t)$ with a suitable function $\varphi \geqslant o$ and constants $\kappa \geqslant 0$, $\delta > 0$. Exploiting the side conditions, in particular, $v(x) = \psi_\lambda(x)\eta$ and $W'(\eta)q = o$, one sees that all assumptions are satisfied if $s(\xi) \in \varphi(\xi)G$ for $\xi \in \bar{\Omega}_0$ and

$$
\begin{gathered}
-\Delta\varphi(\xi) + \sum_{i=1}^{n} v_i(\xi, t)(\partial/\partial\xi_i \varphi(\xi)) - \kappa\varphi(\xi) \geqslant 0 \\
\text{for} \quad \xi \in \Omega_0, t \in (0, T].
\end{gathered}
\tag{5.7}
$$

We collect a few statements which can be obtained by solving these inequalities.

(i) *If $s(\xi) \in G$ $(\xi \in \bar{\Omega}_0)$, then $v(\xi, t) \in G$ $(\xi \in \bar{\Omega}_0, 0 \leqslant t \leqslant T)$ for each solution $v \in R$.*

(ii) *Let $h(r) := \int_r^{r_0} s \exp(\rho s)\, ds$ with constant $\rho \geqslant 0$, $r = \|\xi - \xi_0\|$, $\xi_0 \in \Omega_0$, $r_0 > r$ for $\xi \in \bar{\Omega}_0$. If $s(\xi) \in h(r)G$ and $\|s(\xi)\| \leqslant \rho$ for all $\xi \in \bar{\Omega}_0$, then for each solution $v \in R$*

$$
v(\xi, t) \in \exp(-\kappa t)\, h(r)G \qquad (\xi \in \bar{\Omega}_0, 0 \leqslant t \leqslant T) \qquad \textit{with} \quad \kappa h(0) = n. \tag{5.8}
$$

(iii) *If $s_k(\xi) \geqslant 0$ $(\xi \in \bar{\Omega}_0)$ for some index k, then $v_k(\xi, t) \geqslant 0$ $(\xi \in \bar{\Omega}_0, 0 \leqslant t \leqslant T)$ for each solution $v \in R$.*

(iv) *Let* $h(r) = \frac{1}{2}(r_0^2 - r^2)$ *with* $r = \|\xi - \xi_0\|$, $\xi_0 \notin \Omega_0$, $\xi \leqslant \xi_0$ *and* $r_0 > r$ *for all* $\xi \in \bar{\Omega}_0$. *If* $s(\xi) \in h(r)G$ *and* $s(\xi) \geqslant o$ *for all* $\xi \in \bar{\Omega}_0$, *then* (5.8) *holds for each solution* $v \in R$.

Statement (i) is derived using $\varphi(\xi) \equiv 1$ and $\kappa = 0$. For proving (ii) one defines $\varphi(\xi) = h(r)$ and observes that the sum in (5.7) can be estimated from above by $\|v(\xi, t)\|\ \|\varphi'(\xi)\|$ and that $\|v(\xi, t)\| \leqslant \rho$, according to (i). The calculations are analogous to those in Section 4. Statement (iii) is obtained by using $W(y) = -y_k$ and $\varphi(\xi) \equiv 0$. To prove (iv) one uses the fact that $v(\xi, t) \geqslant o$, according to (iii), and that therefore the sum in (5.7) is non-negative.

Inequality (5.7) was obtained by splitting the conditions on ψ_λ in a specific way. One may also use other splittings. For example, employing the side condition $v = (\psi + \lambda z)\eta$ one can obtain inequalities of the form

$$z_t + \left(\sum_i \eta_i D_i \psi\right) z > 0, \qquad \psi_t - \Delta\psi + \left(\sum_i \eta_i D_i \psi\right) \psi \geqslant 0.$$

Finally, Theorem 3 yields also a uniqueness statement. One transforms the given problem into a problem for the difference $u = w - v$ of two solutions v, w and then applies the theorem with $W(y) = y^T y$ and $\psi_\lambda(\xi, t) = \lambda \exp(\delta t)$. The resulting inequalities for ψ_λ are satisfied if $\delta + \sum_{i=1}^n (\partial/\partial\xi_i)\, v(\xi, t) > 0$. Therefore, one obtains:

(v) *If problem* (5.6) *has a solution* $v \in R$ *with bounded derivatives* $\partial/\partial\xi_i v$ $(i = 1, 2, \ldots, n)$, *then* v *is the only solution in* R.

6. Estimates by Several Functions ψ_ℓ

6.1. *The Theorem*

We consider now more general estimates of the form

$$v(x) \in \psi_\ell(x)\, G_\ell \qquad (x \in \bar{\Omega};\ \ell = 1, 2, \ldots, N) \tag{6.1}$$

which are described by several sets $G_\ell \subset \mathbb{R}^n$ and several functions ψ_ℓ. This allows us also to treat more general operators of the form (3.1), (5.1) where the components of $\mathscr{L}[u]$ and $\bar{\mathscr{L}}[u]$ need not be defined by the same operators L and $\bar{L}$, respectively. We assume now that

$$\mathscr{L}_i[u](x) = L_i[u_i](x) \qquad \text{and} \qquad \bar{\mathscr{L}}_i[u](x) = \bar{L}_i[u_i](x) \tag{6.2}$$

with L_i and $\bar{L}_i$ having the form (3.3) and (5.3), except that all "coefficients" have an index i. For instance,

$$L_i[\varphi](x) = -A_i(x) \cdot \varphi''(x) + b_i(x) \cdot \varphi'(x) \tag{6.3}$$

with symmetric $A_i(x) \geqslant_d O$. We shall write $L_i = L_j$, if the operators L_i and L_j have the same coefficients (i.e., $A_i(x) = A_j(x)$ and $b_i(x) = b_j(x)$ for $x \in \Omega$), and define $\bar{L}_i = \bar{L}_j$ analogously.

To consider coefficients b_i, β_i and γ_i which depend on i constitutes no essential generalization since the corresponding terms may be incorporated into the nonlinear parts of M, without changing the results. The essential point is that under certain conditions the *leading coefficients* $A_i(x)$, $\bar{A}_i(x)$ and $\alpha_i(x)$, too, may be different for different indices i.

To which extend these coefficients may be different depends on the sets G_ℓ. We shall assume that all sets G_ℓ satisfy assumption ($\mathcal{g}$), and denote by W_ℓ (in place of W) the function which describes G_ℓ and by Γ_ℓ the boundary of G_ℓ.

To each index $\ell \in \{1, 2,..., N\}$ there exists a subset P_ℓ of indices $i \in \{1, 2,..., n\}$ such that $i \in P_\ell$ if and only if $\partial/\partial y_i W_\ell(y) \neq 0$ for some $y \in \Gamma_\ell$ (that means $W_\ell(y)$ depends on y_i on Γ_ℓ). Using these sets P_ℓ we can formulate our essential assumption regarding the leading coefficients.

Assumption (A). Suppose that $L_i = L_j$ and $\bar{L}_i = \bar{L}_j$ for all pairs of indices i, j such that $i \in P_\ell$ and $j \in P_\ell$ for some $\ell \in \{1, 2,..., N\}$.

EXAMPLE 4. Suppose that $x \in \Omega$.

(a) In case $N = 1$ and $W(y) = y^T y$, all $A_i(x)$ must be equal.

(b) In case $N = 2n$, $W_\ell(y) = y_\ell$, $W_{n+\ell}(y) = -y_\ell$ ($\ell = 1, 2,..., n$), the matrices $A_i(x)$ may all be different.

(c) Let y be split into two subvectors $y^1 = (y_1,..., y_r)^T$ and $y^2 = (y_{r+1},..., y_n)^T$ with $1 \leqslant r < n$. Then, if $W_1(y) = (y^1)^T y^1$, $W_2(y) = (y^2)^T y^2$, and $N = 2$, all matrices $A_i(x)$ with $i \leqslant r$ must be equal, and all matrices $A_i(x)$ with $i > r$ must be equal. ∎

In each of Sections 3 and 5 we had defined a space $\mathscr{R}$ such that the functions in $\mathscr{R}$ had "sufficient smoothness properties." Since now, however, different smoothness properties may be necessary for different components u_i of the functions u to which M is applied, we may work with different function spaces $R_i \subset C_0(\bar{\Omega})$ such that $u_i \in R_i$ has sufficient smoothness properties. Then we use $R = R_1 \times R_2 \times \cdots \times R_n$ and define $\mathscr{R}_\ell$ to be the intersection of all R_i with $i \in P_\ell$. It seems unnecessary to describe all these spaces in detail. (Of course, one may also define $\mathscr{R}$ to be the intersection of all R_i and $R = \mathscr{R}^n$.)

Suppose that $\psi_\ell \in \mathscr{R}_\ell$ with $\psi_\ell \geqslant o$ and that $\{\mathfrak{z}_{\ell,\lambda}: 0 \leqslant \lambda < \infty\}$ denotes a family of functions in $\mathscr{R}_\ell$ with the properties required of $\mathfrak{z}_\lambda$ in Section 3; then define $\psi_{\ell,\lambda} = \psi_\ell + \mathfrak{z}_{\ell,\lambda}$ ($\ell = 1, 2, ..., N$). Let $v \in R$ denote a fixed function, as before. Moreover, define

$$K_\lambda(x) = \bigcap_{\ell=1}^{N} \psi_{\ell,\lambda}(x)\, G_\ell \qquad \text{and} \qquad K(x) = K_0(x)$$

so that (6.1) is equivalent to $v(x) \in K(x)$ ($x \in \bar{\Omega}$).

Finally, denote by ($\mathscr{j}_\ell$) and ($\mathscr{k}_\ell$) the assumptions which are obtained by changing assumption ($\mathscr{j}$) and ($\mathscr{k}$), respectively, in the following way:

(a) Replace W and ψ_λ by W_ℓ and $\psi_{\ell,\lambda}$.

(b) Replace L, A, $\bar{L}$ and $\bar{A}$ by L_i, A_i, $\bar{L}_i$ and $\bar{A}_i$ with some $i \in P_\ell$.

(c) Add the relation $v(x) \in K_\lambda(x)$ in (3.7) and (5.5) as an additional side condition on $v(x)$, so that, for instance, the modified relations (3.7) assume the form

$$v(x) = \psi_{\ell,\lambda}(x)\eta, \qquad v'(x) = \eta\psi'_{\ell,\lambda}(x) + \psi_{\ell,\lambda}(x)q, \qquad v(x) \in K_\lambda(x). \tag{6.4}$$

Observe that the terms η and q which occur in ($\mathscr{j}_\ell$) and ($\mathscr{k}_\ell$) describe quantities which also depend on ℓ. For instance, $W_\ell(\eta) = 1$ is required, so that the first relation in (6.4) means that $v(x) \in \psi_{\ell,\lambda}(x)\,\Gamma_\ell$.

THEOREM 5. *If assumption* (A) *and assumptions* ($\mathscr{j}_\ell$) *and* ($\mathscr{k}_\ell$) *are satisfied for all* $\ell \in \{1, 2, ..., N\}$, *then the estimate* (6.1) *holds.*

Proof. Let $\lambda \geqslant 0$ denote the smallest number with $v(x) \in K_\lambda(x)$ ($x \in \bar{\Omega}$), and suppose that (6.1) does not hold. Then $\lambda > 0$, and there exists an index ℓ such that λ is also the smallest number with $v(x) \in \psi_{\ell,\lambda}(x)\, G_\ell$ ($x \in \bar{\Omega}$). This statement can be carried to a contradiction by using the arguments in the proofs of Theorems 1 and 3, where now, however, v satisfies also the last restriction in (6.4). ∎

Remark. In Theorem 5, we have assumed that each set G_ℓ is star-shaped with respect to its (interior) point $y^0 = o$. Instead, we may assume that G_ℓ is star-shaped with respect to an arbitrary interior point y^ℓ. Then Theorem 5 can easily be generalized to obtain estimates of the form $v(x) \in \tilde{K}(x) := \bigcap_{\ell=1}^{N} \psi_\ell(x)(G_\ell - y^\ell)$. Observe that $\tilde{K}(x)$ need not be star-shaped.

The shape of the sets $K_\lambda(x)$, in general, will depend on x and λ, as the following example shows.

EXAMPLE 5. *Let* $n = 2$ *and* $N = 5$ *and suppose that the five sets* G_ℓ *are described by, respectively,*

$$y_1^2 + y_2^2 \leqslant 1, \qquad 4y_1 + 1 \geqslant 0, \qquad 4y_1 - 1 \leqslant 0,$$
$$4y_2 + 1 \geqslant 0, \qquad \text{and} \qquad 4y_2 - 1 \leqslant 0.$$

Also define $\psi_{1,\lambda} = 1 + \lambda$ and $\psi_{\ell,\lambda} = 1 + 5\lambda$ for $\ell > 1$. Then $K_0(x)$ is a square lying inside the circle G_1, and for $\lambda > 3$, $K_\lambda(x)$ is the circle $(1+\lambda)\, G_1$.

In this case relations (6.4) cannot be satisfied for $\lambda > 3$ and $\ell > 1$ since $v(x)$ cannot lie on $\psi_{\ell,\lambda}\Gamma_\ell$ if $v(x) \in (1+\lambda)\, G_1$. That means that all assumptions $(\mathscr{j}_\ell)$ and $(\mathscr{k}_\ell)$ with $\ell > 1$ hold for $\lambda > 3$, and that only $(\mathscr{j}_1)$ and $(\mathscr{k}_1)$ need be verified for such values of λ. ∎

Assumption (Λ) is a condition on the operators L_i and $\bar{L}_i$ for given functions W_ℓ, i.e., given sets G_ℓ. This assumption is closely related to the convexity condition (C) on the set G used by Amann [2] (in [2] this set may be the intersection of several sets G_ℓ). The choice of the functions W_ℓ or the sets G_ℓ, however, does not only determine to which extend the leading coefficients may be different, but has also influence on the possible coupling of f, as discussed in the next section.

6.2. *Conditions on the Coupling of M*

The discussion in the preceding section has shown that there are advantages to using estimates with functions $W_\ell(y)$ which do not depend on all components y_i. But there are disadvantages, too. Using such functions restricts the way in which the vector $Mu(x)$ may be coupled with respect to the derivative $u'(x)$. We shall describe this for the special case of an operator M given by (3.1), (3.2), where $\mathscr{L}_i$ is defined by (6.2), (6.3) and assumption (Λ) holds.

The differential inequality in assumption $(\mathscr{j}_\ell)$ contains the terms

$$A_i(x) \cdot qW_\ell''(x)\, q^T \qquad \text{and} \qquad W_\ell'(\eta)\, f(x, \psi_{\ell,\lambda}(x)\eta, \psi_{\ell,\lambda}(x) q + \eta\psi_{\ell,\lambda}'(x))$$

with $i \in P_\ell$. The first term depends only on elements q_{ik} of q with $i \in P_\ell$, and the second term depends only on components f_i with $i \in P_\ell$. Moreover, the elements q_{ik} with $j \notin P_\ell$, which may occur in the last argument of f_i, do not occur in the restriction $W_\ell'(\eta) q = o$.

As a consequence, the component $f_i(x, y, p)$ with $i \in P_\ell$ may depend on elements p_{jk} of p with $j \notin P_\ell$ only in a "very limited way" since otherwise assumption $(\mathscr{j}_\ell)$ cannot be satisfied. For a linear M this statement can be made more precise:

Suppose that f is given by (4.2). *Then assumption* $(\mathscr{j}_\ell)$ *can only be satisfied if the ith row of $B_j(x)$ vanishes (is a null vector) for all $x \in \Omega$ and all $i \in P_\ell$ and $j \notin P_\ell$.*

EXAMPLE 6. Consider the case $W_1(y) = (y^1)^T y^1$, $W_2(y) = (y^2)^T y^2$ treated in Example 4(c), where $y^1 \in \mathbb{R}^r$, $y^2 \in \mathbb{R}^{n-r}$ and $y^T = ((y^1)^T, (y^2)^T)$. For each matrix $q \in \mathbb{R}^{n,m}$ we use an analogous splitting $q^T = ((q^1)^T, (q^2)^T)$

with $q^1 \in \mathbb{R}^{r,m}$, $q^2 \in \mathbb{R}^{n-r,m}$. For simplicity, we assume that $L_1[\varphi] = -\Delta\varphi$ and that $Mv = o$. Then the differential inequality in $(\mathscr{j}_1)$ is equivalent to

$$-\Delta\psi_{1,\lambda} + Q_1\psi_{1,\lambda} + \sum_{i=1}^{r} \eta_i f_i(x, v, v') > 0$$

with $Q_1 = q^1 \cdot q^1$; and the corresponding side conditions can be written as

$$v^1 = \psi_{1,\lambda}\eta^1, \qquad (v^1)' = \eta^1\psi'_{1,\lambda} + \psi_{1,\lambda}q, \qquad \|\eta\| = 1,$$
$$(\eta^1)^T q = o, \qquad \|v^2\| \leqslant \psi_{2,\lambda}.$$

Since there are no conditions on the function $(v^2)'$, which is considered to be unknown, a function $f_i(x, y, p)$ with $i \leqslant r$, in general, must not depend on the vector p^2 at all. Of course, under certain special conditions, the differential inequality can be solved, even if p^2 occurs. For instance, this is the case if each function $f_i(x, y, p)$ with $i \leqslant r$ is globally bounded or each function $y_i f_i(x, y, p)$ is positive. ∎

6.3. *Two-Sided Bounds*

As a special application of Theorem 5 we derive results on two-sided bounds, where again the operator M is given by (3.1), (3.2), (6.2) and (6.3). We choose functions W_ℓ as in Example 4(b), so that all operators L_i may be different. Suppose that ψ_i and $\psi_{i,\lambda} = \psi_i + \mathfrak{z}_{i,\lambda}$ $(i = 1, 2,..., 2n)$ denote functions with the properties required in Section 6.1. Here, $\mathscr{R}$ and $R = \mathscr{R}^n$ may be defined as in Section 3.

We shall use the following notation: $\varphi_i = -\psi_{n+i}$, $\varphi_{i,\lambda} = -\psi_{n+i,\lambda}$ $(i = 1, 2,..., n)$; φ, ψ, φ_λ, and ψ_λ are elements of R with components φ_i, ψ_i, $\varphi_{i,\lambda}$ and $\psi_{i,\lambda}$ $(i = 1, 2,..., n)$, respectively. (The vector-valued function φ_λ must not be confused with a component of φ.)

Corollary 5a. *Suppose that for each* $i \in \{1, 2,..., n\}$ *the following conditions* (ℓ_i), $(\mathscr{j}_i)$ *and* $(\mathscr{j}_{n+i})$ *are satisfied*:

(ℓ_i) $\varphi_i(x) \leqslant v_i(x) \leqslant \psi_i(x)$ *for all* $x \in \partial\Omega$;

$(\mathscr{j}_i)$ $L_i[\psi_{i,\lambda}](x) + f_i(x, v(x), v'(x)) > (Mv)_i(x)$

for all $x \in \Omega$ *and* $\lambda > 0$ *with*

$$v_i(x) = \psi_{i,\lambda}(x), \qquad v_i'(x) = \psi'_{i,\lambda}(x),$$
$$\varphi_{j,\lambda} \leqslant v_j \leqslant \psi_{j,\lambda} \qquad (j = 1, 2,..., n);$$

$(\mathscr{j}_{n+i})$ $L_i[\varphi_{i,\lambda}](x) + f_i(x, v(x), v'(x)) < (Mv)_i(x)$

for all $x \in \Omega$ *and* $\lambda > 0$ *with*

$$v_i(x) = \varphi_{i,\lambda}(x), \qquad v_i'(x) = \varphi_{i,\lambda}'(x),$$

$$\varphi_{j,\lambda} \leqslant v_j \leqslant \psi_{j,\lambda} \qquad (j = 1, 2,..., n).$$

Then $\varphi_i \leqslant v_i \leqslant \psi_i$ $(i = 1, 2,..., n)$.

This result is an immediate consequence of Theorem 5. Observe that the general assumptions on the functions ψ_i require that $\varphi_i \leqslant o \leqslant \psi_i$ $(i = 1, 2,..., n)$ in the above corollary. It can be shown, however, by a simple transformation that this requirement may be replaced by $\varphi_i \leqslant \psi_i$ $(i = 1, 2,..., n)$ (replace u by $\tilde{u} = u - \frac{1}{2}(\varphi + \psi)$).

The above result is formulated for Dirichlet boundary terms (3.2). Analogous results for more general boundary terms can be proved in essentially the same way.

The differential inequality in $(\mathscr{j}_i)$, in general, can only be satisfied if $f_i(x, v(x), v'(x))$ does not depend on any $v_j'(x)$ with $j \neq i$. Again, under suitable boundedness conditions on f_i, there may exist solutions $\psi_{i,\lambda}$ in other cases also. However, the condition that f_i be positive, in general, does not help, since f_i also occurs in $(\mathscr{j}_{n+i})$.

For these reasons, results on two-sided bounds are usually formulated for the case that $f(x, y, p)$ is not coupled with respect to p, i.e., $f_i(x, u(x), u'(x))$ depends only on x, $u(x)$ and $Du_i(x)$. Then the terms $v'(x)$ in $(\mathscr{j}_i)$ and $(\mathscr{j}_{n+i})$ may be replaced by $\psi_\lambda'(x)$ and $\varphi_\lambda'(x)$, respectively. If, in addition, $f(x, y, p)$ is quasiantitone with respect to y (that means, if $f_i(x, y, p)$ is an antitonic function of all y_j with $j \neq i$), then assumptions $(\mathscr{j}_i)$ and $(\mathscr{j}_{n+i})$ together can be replaced by $(M\varphi_\lambda)_i\,(x) < (Mv)_i\,(x) < (M\psi_\lambda)_i\,(x)$ $(x \in \Omega, \lambda > 0)$.

Corollary 5a can also be derived from abstract results on two-sided bounds in [17, 18, p. 259ff.]. Each of the differential inequalities $(\mathscr{j}_i)$ and $(\mathscr{j}_{n+i})$ may be split into two inequalities, as described in [18, pp. 281ff.] for ordinary differential operators.

7. Existence and Estimation

For certain boundary value problems $Mu = r$ with a nonlinear term f not depending on u', the above theory gives rise to a particularly simple proof for the existence of a solution which satisfies an estimate as considered in the previous sections. In Section 7.1 we describe the method of proof for Dirichlet boundary conditions; the results can easily be carried over to other problems. Section 7.2 yields an example.

7.1. *Existence Proof by the Method of Modification*

Let G_ℓ and ψ_ℓ ($\ell = 1, 2, \ldots, N$) be defined as in Section 6, but assume now that the sets G_ℓ are convex, and the functions ψ_ℓ belong to $C_3(\bar{\Omega})$ and satisfy

$$\psi_\ell(x) > 0 \qquad (x \in \bar{\Omega}). \tag{7.1}$$

Suppose that M is given by (3.1), (3.2), (6.2) and (6.3) with assumption (Λ) being satisfied, and assume that the function f does not depend on p: $f(x, y, p) = f(x, y)$. We then consider a problem of the form

$$Mu(x) = o \qquad \text{for } x \in \Omega, \qquad u(x) = s(x) \qquad \text{for } x \in \partial\Omega, \tag{7.2}$$

which can be written as $Mu = r$ with $r(x) = o$ for $x \in \Omega$ and $r(x) = s(x)$ for $x \in \partial\Omega$.

We assume that the occurring quantities satisfy the following conditions: Ω is of class C_3, $A_i \in C_1(\bar{\Omega}, \mathbb{R}^{m,m})$, $A_i(x)$ is symmetric and positive definite for each $x \in \bar{\Omega}$, $b_i \in C_1(\bar{\Omega}, \mathbb{R}^{1,m})$, f is a continuously differentiable mapping of $\Omega \times \mathbb{R}^n$ into $\mathbb{R}^n$, and $s \in C_3(\partial\Omega)$. (These assumptions can slightly be relaxed by using Hölder conditions.)

Now we define $\mathscr{R} = C_2(\bar{\Omega})$ and $R = \mathscr{R}^n$, and we want to prove that problem (7.2) has a solution $u^* \in R$ which satisfies

$$u^*(x) \in K(x) := \bigcap_{\ell=1}^{N} \psi_\ell(x)\, G_\ell \qquad \text{for} \quad x \in \bar{\Omega}.$$

For this purpose we assume that there exists a bounded set $\hat{K} \notin \mathbb{R}^n$ such that $K(x) \subset \hat{K}(x \in \bar{\Omega})$. Obviously, the boundary $\partial K(x)$ of $K(x)$ is the union of the sets $\partial_\ell K(x) := K(x) \cap \psi_\ell(x)\, \Gamma_\ell$ ($\ell = 1, 2, \ldots, N$).

THEOREM 6. *Suppose that $s(x) \in K(x)$ for $x \in \partial\Omega$, and that for each $\ell \in \{1, 2, \ldots, N\}$ the differential inequality*

$$\omega_\ell(\eta)\, L_i[\psi_\ell](x) + W_\ell'(\eta)\, f(x, \psi_\ell(x)\eta) \geqslant 0$$

holds for some $i \in P_\ell$ and all $x \in \Omega$ and $\eta \in \mathbb{R}^n$ which satisfy $W_\ell(\eta) = 1$, $\psi_\ell(x)\eta \in \partial_\ell K(x)$.

Then the equation $Mu = r$ has a solution $u^* \in C_2^n(\bar{\Omega})$ such that $u^*(x) \in K(x)$ ($x \in \bar{\Omega}$).

Proof. The statements are proved by applying the *method of modification*. We define a *modified problem*, prove that this problem has a solution u^* and finally use Theorem 5 in order to show that u^* is also a solution of the given problem.

(1) For $u \in C_0^n(\bar{\Omega})$ we define a truncation operation $u \to u^{\#} \in C_0^n(\bar{\Omega})$ in the following way. If $u(x) \in K(x)$, then $u^{\#}(x) = u(x)$; if $u(x) \notin K(x)$, then

$u^{\#}(x) = tu(x)$ with the (uniquely determined) value t for which $tu(x) \in \partial K(x)$.

Then a modified operator $M^{\#} = A - B$ on R is defined by

$$Au(x) = \mathscr{L}[u](x) + cu(x), \qquad Bu(x) = cu^{\#}(x) - f(x, u^{\#}(x)) \qquad \text{for } x \in \Omega,$$

$$Au(x) = u(x), \qquad Bu(x) = o \qquad \text{for } x \in \partial\Omega,$$

where $c > 0$ denotes a constant such that

$$L_i[\psi_\ell](x) + c\psi_\ell(x) > 0 \qquad \text{for } x \in \Omega,\ \ell \in \{1, 2, ..., N\},\ i \in P_\ell. \tag{7.3}$$

(2) The modified problem $M^{\#}u = r$ can be transformed into an equivalent fixed-point equation $u = Tu$ in $X = C_0^n(\bar{\Omega})$ with $Tu = g + \mathscr{K}Bu$, where the occurring quantities have the following meaning. The operator $\mathscr{K}$ is the extension onto X of the solution operator $\tilde{K}$: $C_\mu^n(\bar{\Omega}) \to C_{2+\mu}^n(\Omega)$ of the boundary value problem $Au(x) = w(x)$ $(x \in \Omega)$, $u(x) = o$ $(x \in \partial\Omega)$, with some $\mu \in (0, 1)$, and g is the solution of $Au(x) = o$ $(x \in \Omega)$, $u(x) = s(x)$ $(x \in \partial\Omega)$. The operator $\mathscr{K}$ maps X compactly into any space $C_\sigma^n(\bar{\Omega})$ with $\sigma \in [0, 2)$. (For details see Amann [1]; the results presented there can here be applied to each component of Au.) Due to these properties of $\mathscr{K}$ and the definition of B, the operator T on X is continuous and maps X into a relatively compact subset of this space. Thus, the operator T has a fixed point $u^* \in X$, according to Schauder's fixed-point theorem.

In verifying the equivalence of the fixed-point equation, one uses the fact that $cu^{\#} - f(x, u^{\#})$ satisfies a Lipschitz condition if $u(x) \in C_1^n(\bar{\Omega})$. Observe that for $u(x) \neq o$ we have $u^{\#}(x) = \sigma(x) \|u(x)\|^{-1} u(x)$, where $\sigma(x)$ is the infimum of all values $\sigma_\ell(x) = \rho_\ell(x) \|\eta_\ell(x)\|$ with $\rho_\ell(x) = \inf\{\rho_\ell(x), \psi_\ell(x)\}$, for which $u(x) = \rho_\ell(x)\, \eta_\ell(x)$ with $\rho_\ell(x) > 0$ and $W(\eta_\ell(x)) = 1$ (compare (3.9)).

(3) Finally, Theorem 5 is applied to $M^{\#}$ and u^* in place of M and v in order to show that $u^*(x) \in K(x)$ $(x \in \bar{\Omega})$ and hence $Mu^* = M^{\#}u^* = r$. We choose $\psi_{\ell,\lambda} = (1 + \lambda)\, \psi_\ell$ $(\ell = 1, 2, ..., N)$. Then assumption $(\mathscr{E}_\ell)$ holds and assumption $(\mathscr{J}_\ell)$ requires that

$$\omega_\ell(\eta)(1 + \lambda)\, L_i[\psi_\ell](x) + W_\ell'(\eta)[cu(x) - cu^{\#}(x) + f(x, u^{\#}(x))] > 0 \tag{7.4}$$

for some $i \in P_\ell$ and all $x \in \Omega$, $\lambda > 0$, $\eta \in \Gamma_\ell$ satisfying $u(x) = (1 + \lambda)\, \psi_\ell(x)\eta$, $u(x) \in (1 + \lambda)\, K(x)$. If $u(x)$ has these properties, then $\psi_\ell(x)\eta \in \partial K(x)$ and hence $u^{\#}(x) = \psi_\ell(x)\eta$. Therefore, condition $(\mathscr{J}_\ell)$ is satisfied, due to (7.3) and the differential inequality required in this theorem. ∎

The above assumptions may be relaxed and the proof may be modified in many ways. Some of the assumptions are of a more technical nature. For instance, one can often avoid the requirement (7.1) by defining $u^{\#}$ and $\psi_{\ell,\lambda}$ differently. For example, *if all functions* ψ_ℓ $(\ell = 1, 2, ..., N)$ *are equal to a* *function* $\psi \geqslant o$ *in* $\mathscr{R}$, *then assumption* (7.1) *can be omitted.* This can be seen by choosing $\psi_{\ell,\lambda} = \psi + \lambda$ in the proof of the theorem. In other cases one may choose $u^{\#}(x)$ to be the point in $K(x)$ which is closest to $u(x)$. Moreover, the results and the method of proof can be carried over to other boundary conditions. Nonlinear boundary conditions can also be treated.

Finally, let us consider the special case of linear functions

$$W_\ell(y) = w_\ell\, y \qquad \text{with} \quad w_\ell \in \mathbb{R}^{1,n} \ (\ell = 1, 2, ..., N).$$

Here $K(x)$ is the polyhedral convex set of all $y \in \mathbb{R}^n$ with $w_\ell\, y \leqslant \psi_\ell(x)$ $(\ell = 1, 2, ..., N)$, and $\partial_\ell K(x)$ is the side of $K(x)$ which is described by $w_\ell\, y = \psi_\ell(x)$. The assumptions of Theorem 6 can now be written in the following form:

$$w_\ell\, s(x) \leqslant \psi_\ell(x) \qquad \text{for} \quad x \in \partial\Omega, \tag{7.5}$$

$$L_i[\psi_\ell](x) + w_\ell\, f(x, y) \geqslant 0 \tag{7.6}$$

for some $i \in P_\ell$ and all $x \in \Omega$, $y \in \mathbb{R}^n$ with $y \in \partial_\ell K(x)$.

In this case assumption (7.1) may be droped if $K(x)$ has interior points for all $x \in \bar{\Omega}$. This is seen by applying a simple transformation $\tilde{y} = y - h(x)$, where $h \in C_3^n(\bar{\Omega})$ and $h(x) \in K(x)$. The transformed functions $\tilde{\psi}_\ell = \psi_\ell - w_\ell\, h$ satisfy $\tilde{\psi}_\ell \geqslant o$; moreover, $\tilde{\psi}_\ell(x) > 0$ if $h(x)$ is an interior point of $K(x)$.

In the case $\psi_\ell(x) \equiv 1$ $(\ell = 1, 2, ..., N)$ the result of Theorem 6 essentially is known (see [2, 15, 25] and compare also [11]). Weinberger's result [25, Theorem 3] is formulated differently. This author defines also a modified problem (using the projection of $u(x)$ on $K = K(x)$) and states that a solution of the given problem has values in K, if the modified problem has at least one solution and the given problem has at most one solution. Theorem 4 in [25] contains conditions such that the values of the solution lie on ∂K.

7.2. *An Example*

To illustrate the above results we consider the following boundary value problem on a bounded domain Ω (with boundary of class C_3) for a function $c \in \mathscr{R}^4$, $\mathscr{R} = C_0(\bar{\Omega}) \cap C_2(\Omega)$:

$$-\Delta c(x) + B^T g(c(x)) = o \qquad \text{for} \quad x \in \Omega, \tag{7.7}$$

$$c(x) = s \qquad \text{for} \quad x \in \partial\Omega \tag{7.8}$$

with $s^T = (2, 2, 0, 3)$ and

$$B = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix},$$

$$g(c) = \begin{pmatrix} g_1(c) \\ g_2(c) \end{pmatrix}, \qquad g_1(c) = c_1 c_2 - c_3, \qquad g_2(c) = c_1 c_3 - c_4.$$

The problem describes the steady state of a very simple chemical reaction; the quantities c_i are concentrations, and g_1, g_2 are reaction rates. We are only interested in solutions with $c_i(x) > 0$ for $x \in \Omega$.

In the following we shall make use of the special structure of the given system, which will allow us to reduce the number of variables. This structure is typical for certain chemical problems (see [3], for example), so that the methods described below can be carried over to more general cases.

The given problem for $c \in \mathscr{R}^4$ is equivalent to the following problem for $u = \binom{u_1}{u_2} \in R := \mathscr{R}^2$:

$$-\Delta u(x) + f(u(x)) = o \qquad \text{for} \quad x \in \Omega, \tag{7.9}$$

$$u(x) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \text{for} \quad x \in \partial\Omega \tag{7.10}$$

with $f_1(u) = g_1(c)$ and $f_2(u) = -g_2(c)$, where the variables c_ℓ and u_i satisfy

$$u_1 = c_2, \qquad u_2 = c_4;$$

$$c_\ell = \alpha_\ell - w_\ell u, \qquad (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (3, 0, 5, 0), \tag{7.11}$$

$$w_1 = (-1, 1), \qquad w_2 = (-1, 0), \qquad w_3 = (1, 1), \qquad w_4 = (0, -1).$$

The equivalence is established by proving that the functions $u_3 = c_1 + c_3 + 2c_4$ and $u_4 = c_2 + c_3 + c_4$ satisfy the Laplace differential equation and assume the values 8 and 5, respectively, on the boundary, so that the *conservation laws* $u_3(x) \equiv 8$ and $u_4(x) \equiv 5$ hold. (For non-constant boundary values, the solutions u_3 and u_4 of the Laplace equation could be estimated by estimating their known values on $\partial\Omega$.) We note that the conservation laws are equivalent to

$$b_1 c(x) \equiv 3 \qquad \text{and} \qquad b_2 c(x) \equiv 2 \qquad \text{with}$$

$$b_1 = (1, -1, 0, 1),\ b_2 = (-1, 2, 1, 0).$$

The conservation laws yield upper bounds for the concentrations $c_\ell \geqslant o$ if lower bounds are known. In particular, one obtains an a priori estimate $(u_1, u_2) = (c_2, c_4) \leqslant (5, 4)$, which could be used in applying Theorem 5 to problem (7.9), (7.10).

Here, we shall apply the results of Section 7.1 in order to show that the transformed problem (7.9), (7.10) has a solution u^* such that the corresponding solution c^* of the given problem satisfies

$$\varphi(x) \leqslant c^*(x) \qquad \text{for} \quad x \in \bar{\Omega} \tag{7.12}$$

with a lower bound $\varphi \geqslant o$ in $C_3^4(\bar{\Omega})$ to be calculated. This estimate is equivalent to

$$w_\ell u^*(x) \leqslant \psi_\ell(x) := \alpha_\ell - \varphi_\ell(x) \qquad (x \in \bar{\Omega}; \ell = 1, 2, 3, 4),$$

or briefly, $u^*(x) \in K(x)$ $(x \in \bar{\Omega})$. Here, $K(x)$ is a non-rectangular quadrangle with non-empty interior,if

$$b_1 \varphi(x) > 0 \qquad \text{and} \qquad b_2 \varphi(x) > 0. \tag{7.13}$$

(If $b_1 \varphi(x) = b_2 \varphi(x) = 0$, the set $K(x)$ consists of a single point.)

To prove the existence of a solution c^* satisfying (7.13), we have to determine functions ψ_ℓ such that (7.5) and (7.6) hold. According to the remarks at the end of Section 7.1, we need not require (7.1), if (7.13) is satisfied for all $x \in \bar{\Omega}$.

Conditions (7.5) and (7.6) are equivalent to the following requirements on the functions φ_ℓ $(\ell = 1, 2, 3, 4)$:

$$-\Delta \varphi_\ell(x) + F_\ell(u(x)) \leqslant 0 \qquad \text{for} \quad x \in \Omega, u(x) \in \partial_\ell K(x), \tag{7.14}$$

$$\varphi_\ell(x) \leqslant s_\ell \qquad \text{for} \quad x \in \partial\Omega. \tag{7.15}$$

In these formulas $F_\ell(u(x))$ denotes the ℓth component of $B^T g(c(x))$ with $c(x) \in \mathbb{R}^4$ expressed in terms of $u(x) \in \mathbb{R}^2$ by use of (7.11). Moreover, $\partial_\ell K(x)$ denotes the side of the quadrangle $K(x)$ described by $c_\ell(x) = \varphi_\ell(x)$. For example, the differential inequality for φ_4 assumes the form

$$\begin{aligned} &-\Delta \varphi_4 - c_1 c_3 + \varphi_4 \leqslant 0 \\ &\qquad \text{for} \quad c_1 = 3 + c_2 - \varphi_4,\ c_3 = -c_2 - \varphi_4,\ \varphi_2 \leqslant c_2 \leqslant 5 - \varphi_3 - \varphi_4. \end{aligned}$$

Here, $-c_1 c_3$ is a quadratic function of c_2. We shall replace this function by its maximum on $\partial_4 K(x)$, which assumes the value $(\varphi_3 + 2\varphi_4 - 8)\varphi_3$, if $\varphi_2 \geqslant 1$.

Treating all four inequalities (7.14) in an analogous way, one obtains a (vector-valued) differential inequality $-\Delta \varphi(x) + \mathscr{F}(\varphi(x)) \leqslant o$ $(x \in \Omega)$ for φ, together with two side conditions $\varphi_1(x) \geqslant 1$, $\varphi_2(x) \geqslant 1$ $(x \in \Omega)$, where $\mathscr{F} = \mathscr{F}(\varphi)$ has the components:

$$\mathscr{F}_1 = \varphi_1^2 + (2 - \varphi_2)\varphi_1 + \varphi_2 - 5, \qquad \mathscr{F}_2 = \varphi_2^2 + (4 - \varphi_4)\varphi_2 + \varphi_4 - 5,$$

$$\mathscr{F}_3 = \tfrac{3}{2}(1 + \varphi_1)\varphi_3 - \tfrac{1}{2}(1 + \varphi_1)\varphi_1 - 4, \qquad \mathscr{F}_4 = (1 + 2\varphi_3)\varphi_4 + (\varphi_3 - 8)\varphi_3.$$

If these inequalities have a solution $\varphi \geqslant o$ *such that* (7.15) *holds and* (7.13) *is satisfied for all* $x \in \bar{\Omega}$, *then the given problem* (7.7), (7.8) *has a solution* c^* *with property* (7.12).

We remark that *both signs* $<$ *in* (7.13) *may be replaced by* $\leqslant$. This can be seen by modifying the proof in Section 7.1 such that (7.1) need not be required. (Use $\psi_{\ell,\lambda} = \psi_\ell + \lambda$ and define $u^{\#}(x)$ to be the point in $K(x)$ which is closest to $u(x)$.) On the other hand, requiring the stronger properties (7.13) practically causes no difficulty, as seen in the numerical example below.

Using the conservation laws one verifies that $-\Delta c(x) + \mathscr{F}(c(x)) = o$ $(x \in \Omega)$ for each solution c of the given problem (7.7), (7.8). That means the procedure has lead us to a reformulation of the problem. Of course, one could estimate the term $F_\ell(u(x))$ in (7.14) in a less precise way. For example, in estimating $-c_1 c_3$ on $\partial_4 K(x)$ one may replace each factor c_1 and c_3 by its maximum, thus obtaining $\tilde{\mathscr{F}}_4 = (1 + \varphi_3)\varphi_4 - (3 + \varphi_2)\varphi_3 = \mathscr{F}_4 + (\varphi_2 + \varphi_3 + \varphi_4 - 5)\varphi_3 \geqslant \mathscr{F}_4$. Again, each solution c satisfies the corresponding differential inequalities for φ as equations, i.e., with $\leqslant$ replaced by $=$. This observation suggests the possibility of another approach, in which the given equations are first transformed in a "suitable" way and then treated directly without using $u \in \mathscr{R}^2$.

In [18] a method of error estimation has been described, which can be carried over to the case considered here. One calculates an approximate solution Φ with a small *defect* $d[\Phi] = \Delta\Phi - \mathscr{F}(\Phi)$ and tries to find a function $\omega \geqslant o$ such that $\varphi = \Phi - \omega$ satisfies all required conditions. Then one obtains a differential inequality for ω of the form

$$-\Delta\omega(x) + A(\Phi(x))\,\omega(x) + \not{f}(\omega(x)) \geqslant -d[\Phi](x) \qquad (x \in \Omega) \quad (7.16)$$

with $A(\Phi(x)) \in \mathbb{R}^{4,4}$ and a nonlinear term $\not{f}(\omega)$ (such that $\not{f}(\alpha\omega) = \alpha^2 \not{f}(\omega)$ for $\alpha \in \mathbb{R}$). One proves that $A(\Phi(x))z \geqslant o$ $(x \in \bar{\Omega})$ for $z = (4, 6, 5, 20)^T$, if $b_1\Phi(x) \leqslant 3$, $b_2\Phi(x) \leqslant 2$ and $b_3\Phi(x) \geqslant 2$ with $b_3 = (0, 0, 5, 1)$ $(x \in \bar{\Omega})$. Thus, if these inequalities hold and if a constant lower bound γ for $d[\Phi](x)$ is known, one can easily solve the inequality for ω by calculating a constant $\delta \geqslant 0$ such that $\omega(x) = \delta(r_0^2 - r^2)z$ satisfies $-\Delta\omega(x) + \not{f}(\omega(x)) \geqslant \gamma$ $(x \in \Omega)$, where $r = \|x - x_0\|$, $x_0 \in \Omega$ and r_0 is sufficiently large. Observe that here the inequalities (7.13) hold, if Φ satisfies the conservation laws and $r_0^2 - r^2 > 0$ on $\bar{\Omega}$.

Let us finally consider a simple special case, where $\Omega = \{x \in \mathbb{R}^3 : \|x\| \leqslant \frac{1}{2}\}$. We shall not apply a systematic procedure for computing an approximate solution, but simply choose $\Phi = \alpha_0 + \sum_{j=1}^{5} \alpha_j (r^2)^j$ with α_j $(j \neq 0)$ being the coefficients of the Taylor expansion of a solution c^*, and α_0 chosen such that Φ satisfies the given boundary conditions. The functions Φ_ℓ satisfy also the conservation laws. Here, inequality (7.16) holds for $\omega(x) = \delta(r_0^2 - r^2)z$ with $\delta = 3.46 \times 10^{-8}$, $r_0 = 0.5 + \varepsilon$, and each sufficiently small $\varepsilon > 0$. Moreover, all other required conditions are also satisfied. (To choose $\varepsilon > 0$ instead of $\varepsilon = 0$ is only necessary for verifying (7.13) on $\partial\Omega$.)

Consequently, the given problem (7.7), (7.8) has a solution c^* such that

$$c^*(x) \geqslant \Phi(x) - \delta(\tfrac{1}{4} - \|x\|^2)(4, 6, 5, 20)^T \; (x \in \bar{\Omega})$$
$$\text{with } \delta = 3.48 \times 10^{-8}.$$

These inequalities can also be used for obtaining upper bounds.

The crucial point in this estimation procedure is the calculation of a lower bound γ for the defect on the entire domain Ω. In our simple example, the defect is a polynominal (in r^2) and thus can be estimated by calculating values at a finite number of points (see [6] for details).

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